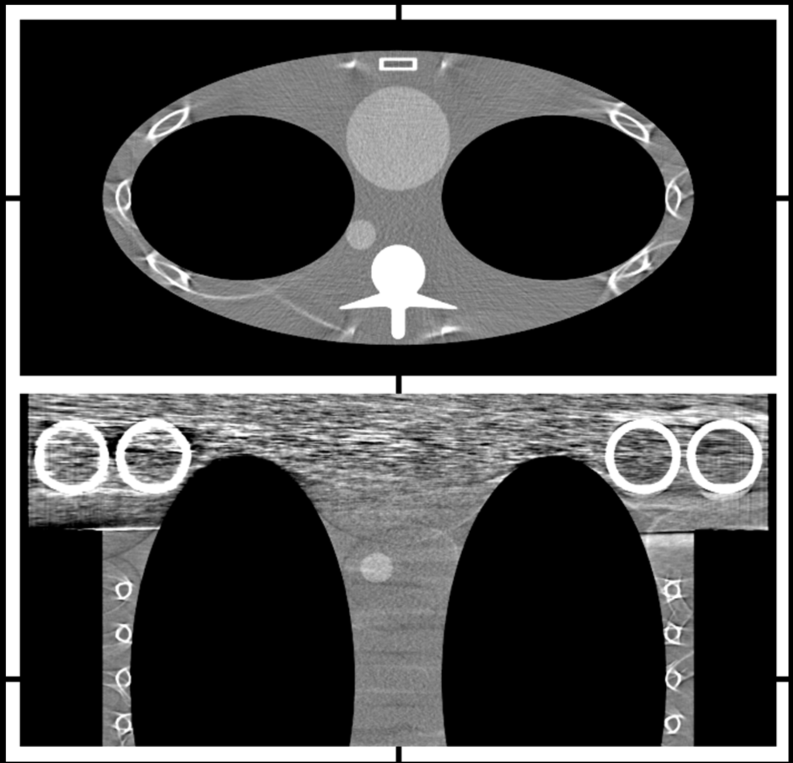
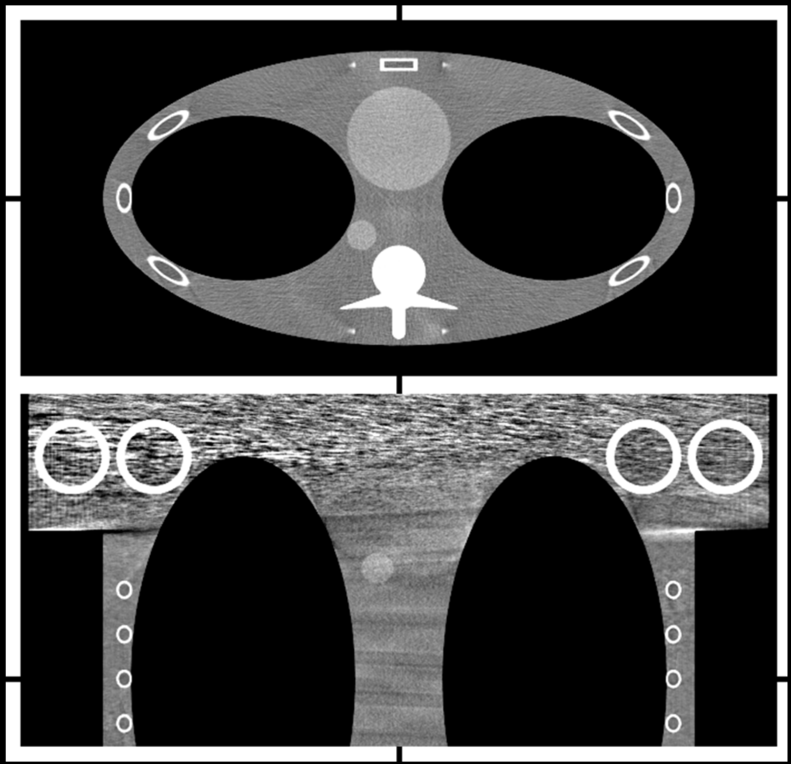
• Spiral
• ASSR Std
• p = 1.0



• Spiral
• EPBP Std
• p = 1.0



• 256 slices
• (0/300)

Kachelrieß et al., Med. Phys. 31(6): 1623-1641, 2004

Advantages of Cone-Beam Spiral CT

- Image quality nearly independent of pitch
- Increase
  - of scan speed
  - of z-resolution
- New applications
  - CT angiography
  - dynamic studies
  - virtual endoscopy
  - cardiac CT
  - DECT
  - ...

Today, complete anatomical regions are routinely scanned with cone-beam spiral CT within a few seconds with isotropic sub-millimeter spatial resolution.

Iterative Image Reconstruction

$$x^2 = y$$

~~$x = \sqrt{y}$~~

Model

$$(x_n + \Delta x_n)^2 = y$$

$$x_n^2 + 2x_n\Delta x_n + \cancel{\Delta x_n^2} = y$$

$$x_n^2 + 2x_n\Delta x_n \approx y$$

$$\Delta x_n = \tfrac{1}{2}(y - x_n^2)/x_n$$

$$x_{n+1} = x_n + \Delta x_n$$

Update equation

Modified from Johan Nuyts, „New image reconstruction techniques", ECR 2012

Influence of Update Equation and Model

$0.5\,(3 - x_n^2)/x_n$	$0.4\,(3 - x_n^2)/x_n$	$0.5\,(3 - x_n^{2.1})/x_n$
$x_0 = 1.$	$x_0 = 1.$	$x_0 = 1.$
$x_1 = 2.$	$x_1 = 1.8$	$x_1 = 2.$
$x_2 = 1.75$	$x_2 = 1.74667$	$x_2 = 1.67823$
$x_3 = 1.73214$	$x_3 = 1.73502$	$x_3 = 1.68833$
$x_4 = 1.73205$	$x_4 = 1.73265$	$x_4 = 1.68723$
$x_5 = 1.73205$	$x_5 = 1.73217$	$x_5 = 1.68734$
$x_6 = 1.73205$	$x_6 = 1.73207$	$x_6 = 1.68733$
$x_7 = 1.73205$	$x_7 = 1.73206$	$x_7 = 1.68733$
$x_8 = 1.73205$	$x_8 = 1.73205$	$x_8 = 1.68733$

$$x^2 = 3, \quad x_0 = 1, \quad x_{n+1} = x_n + \Delta x_n$$

Analytical Reconstruction

1. Problem

$$p(\vartheta,\xi) = \int dxdy\, f(x,y)\delta(x\cos\vartheta + y\sin\vartheta - \xi)$$

2. Solution

$$f(x,y) = \int_0^\pi d\vartheta\, p(\vartheta,\xi) * k(\xi)\Big|_{\xi = x\cos\vartheta + y\sin\vartheta}$$

3. Discretisation

$$\boldsymbol{f} = \boldsymbol{R}^{\mathrm{T}} \cdot \boldsymbol{K} \cdot \boldsymbol{p} = \boldsymbol{R}^{\mathrm{T}} \cdot (\boldsymbol{k} * \boldsymbol{p})$$

Classical Iterative Reconstruction

1. Problem

$$p(\vartheta,\xi) = \int dxdy\, f(x,y)\delta(x\cos\vartheta + y\sin\vartheta - \xi)$$

2. Discretisation

$$\boldsymbol{p} = \boldsymbol{R} \cdot \boldsymbol{f}$$

3. Solution

$$\boldsymbol{f}_{\nu+1} = \boldsymbol{f}_\nu + \boldsymbol{R}^{\mathrm{T}} \cdot \frac{\boldsymbol{p} - \boldsymbol{R} \cdot \boldsymbol{f}_\nu}{\boldsymbol{R}^2 \cdot \boldsymbol{1}}$$

Linear System and CT System Matrix

$$\underbrace{\boldsymbol{R}}_{\text{Radon or x-ray transform}} \cdot \underbrace{\boldsymbol{f}}_{\text{image to be reconstructed}} = \underbrace{\boldsymbol{p}}_{\text{measured rawdata}}$$

$$\boldsymbol{R} = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1M} \\ r_{21} & r_{22} & \cdots & r_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ r_{N1} & r_{N2} & \cdots & r_{NM} \end{pmatrix}, \boldsymbol{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_M \end{pmatrix}, \boldsymbol{p} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{pmatrix}$$

Kaczmarz‘s Method

$$\underbrace{\boldsymbol{R}}_{N \times M} \cdot \underbrace{\boldsymbol{f}}_{M \times 1} = \underbrace{\boldsymbol{p}}_{N \times 1}$$

$$\boldsymbol{R} = \begin{pmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \vdots \\ \boldsymbol{r}_N \end{pmatrix}, \quad |\boldsymbol{r}_n| = 1$$

$$\boldsymbol{r}_n \cdot \boldsymbol{f} = p_n$$

Kaczmarz‘s Method (2)

- Successively solve $\boldsymbol{r}_n \cdot \boldsymbol{f} = p_n$
- To do so, project onto the hyperplanes

$$\boldsymbol{r}_n \cdot \left(\boldsymbol{f} + \lambda \boldsymbol{r}_n\right) = p_n$$

$$\lambda = p_n - \boldsymbol{r}_n \cdot \boldsymbol{f}$$

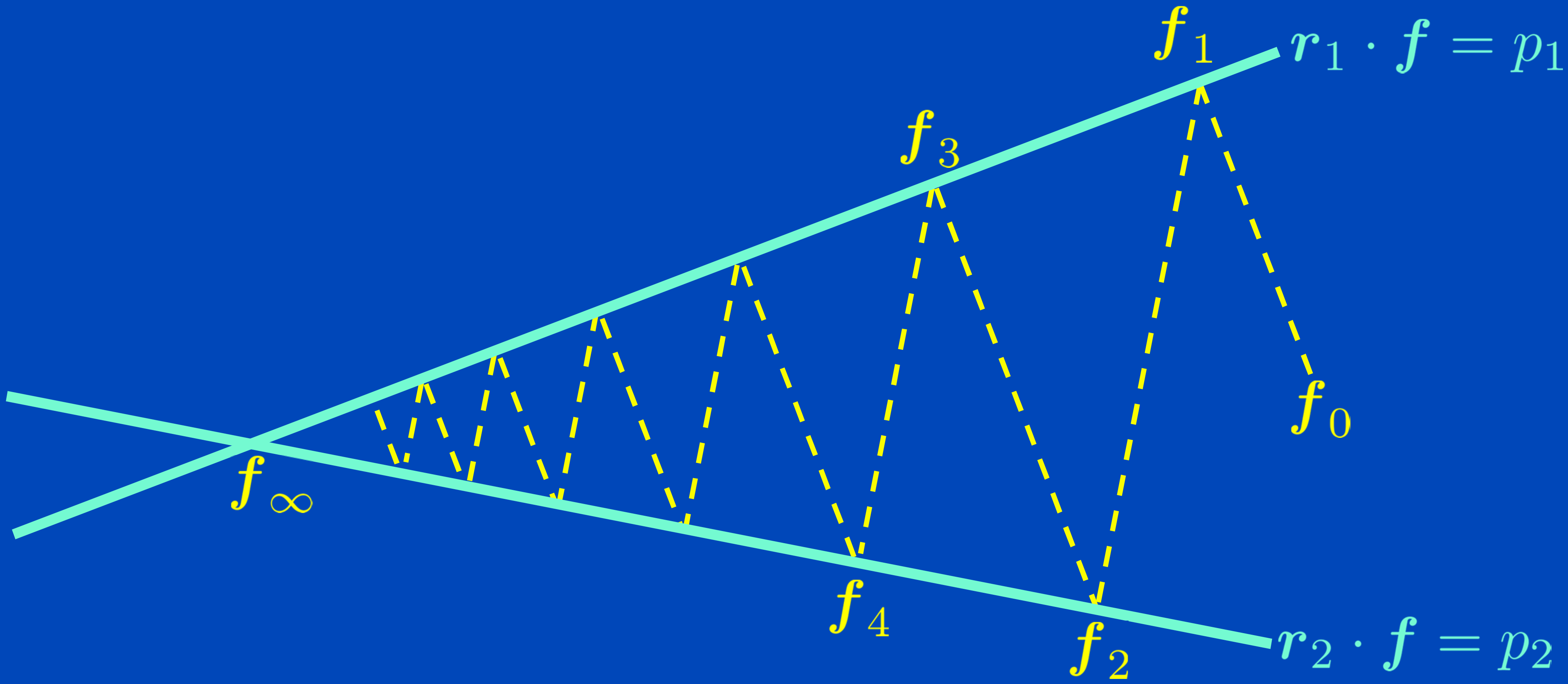
$$\boldsymbol{f}_{\text{new}} = \boldsymbol{f} + \lambda \boldsymbol{r}_n$$

$$\boldsymbol{f}_{\text{new}} = \boldsymbol{f} + \boldsymbol{r}_n \left(p_n - \boldsymbol{r}_n \cdot \boldsymbol{f}\right)$$

- Repeat until some convergence criterion is reached

$$\boldsymbol{f}_{\nu+1} = \boldsymbol{f}_\nu + \boldsymbol{r}_n \left(p_n - \boldsymbol{r}_n \cdot \boldsymbol{f}_\nu\right)$$

Kaczmarz‘s Method (3)



$$\boldsymbol{f}_{\nu+1} = \boldsymbol{f}_\nu + \boldsymbol{r}_n \big(p_n - \boldsymbol{r}_n \cdot \boldsymbol{f}_\nu \big)$$

Kaczmarz in Image Reconstruction: Algebraic Reconstruction Technique (ART)

$$\boldsymbol{f}_{\nu+1} = \boldsymbol{f}_\nu + \boldsymbol{r}_n \big(p_n - \boldsymbol{r}_n \cdot \boldsymbol{f}_\nu \big)$$

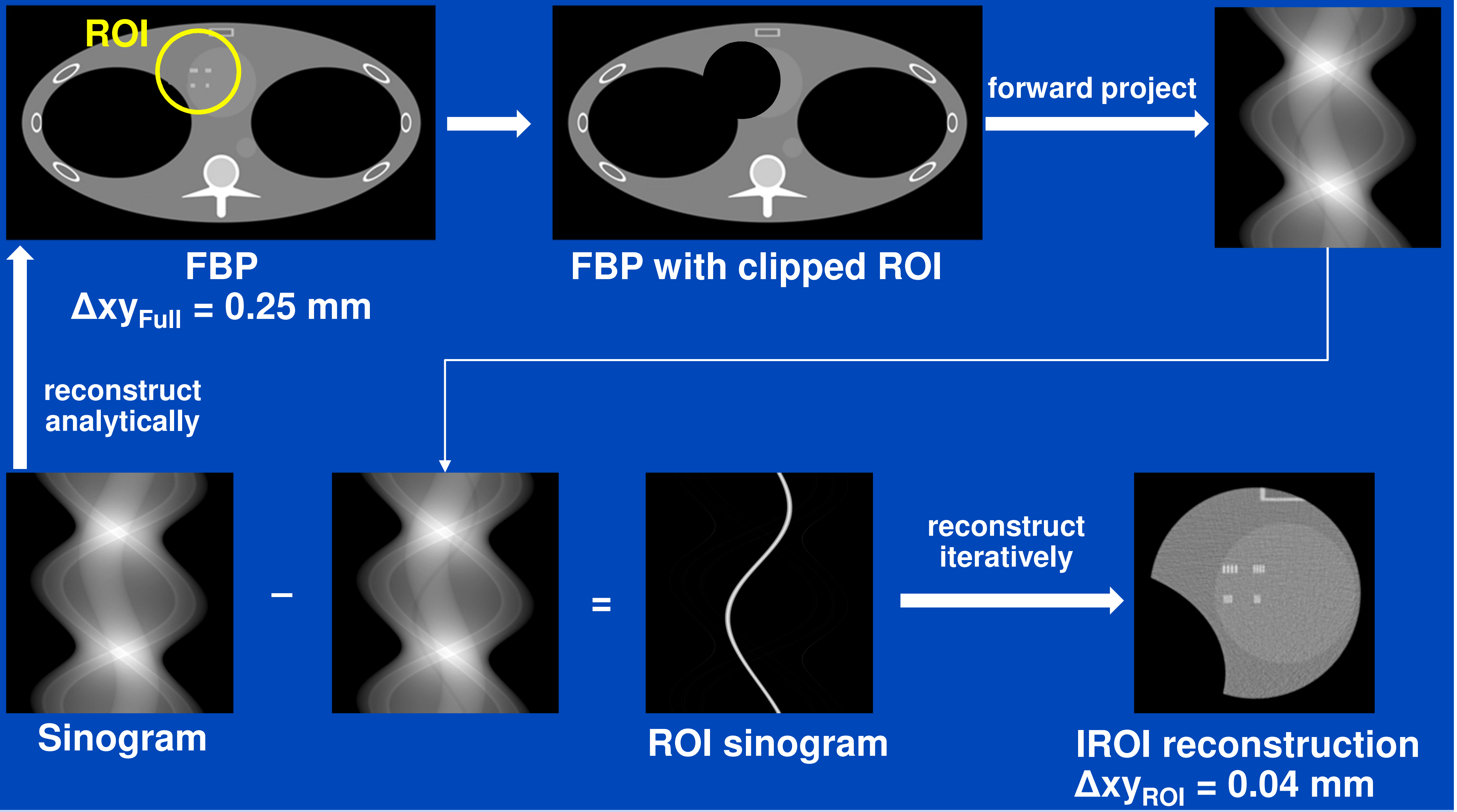
$$\boldsymbol{f}_{\nu+1} = \boldsymbol{f}_\nu + \boldsymbol{R}^{\mathrm{T}} \cdot \frac{\boldsymbol{p} - \boldsymbol{R} \cdot \boldsymbol{f}_\nu}{\boldsymbol{R}^2 \cdot \mathbf{1}}$$

dkfz.

Flavours of Iterative Reconstruction

- **ART** $$\boldsymbol{f}_{\nu+1} = \boldsymbol{f}_\nu + \boldsymbol{R}^{\mathrm{T}} \cdot \frac{\boldsymbol{p} - \boldsymbol{R} \cdot \boldsymbol{f}_\nu}{\boldsymbol{R}^2 \cdot \boldsymbol{1}}$$
- **SART** $$\boldsymbol{f}_{\nu+1} = \boldsymbol{f}_\nu + \frac{1}{\boldsymbol{R}^{\mathrm{T}} \cdot \boldsymbol{1}} \boldsymbol{R}^{\mathrm{T}} \cdot \frac{\boldsymbol{p} - \boldsymbol{R} \cdot \boldsymbol{f}_\nu}{\boldsymbol{R} \cdot \boldsymbol{1}}$$
- **MLEM** $$\boldsymbol{f}_{\nu+1} = \boldsymbol{f}_\nu \frac{\boldsymbol{R}^{\mathrm{T}} \cdot \left(e^{-\boldsymbol{R} \cdot \boldsymbol{f}_\nu}\right)}{\boldsymbol{R}^{\mathrm{T}} \cdot \left(e^{-\boldsymbol{p}}\right)}$$
- **OSC** $$\boldsymbol{f}_{\nu+1} = \boldsymbol{f}_\nu + \boldsymbol{f}_\nu \frac{\boldsymbol{R}^{\mathrm{T}} \cdot \left(e^{-\boldsymbol{R} \cdot \boldsymbol{f}_\nu} - e^{-\boldsymbol{p}}\right)}{\boldsymbol{R}^{\mathrm{T}} \cdot \left(e^{-\boldsymbol{R} \cdot \boldsymbol{f}_\nu} \boldsymbol{R} \cdot \boldsymbol{f}_\nu\right)}$$
- **and dozens more …**

Iterative Region of Interest (IROI)



A. Ziegler, T. Nielsen, and M. Grass, "Iterative Reconstruction of Region of Interest for Transmission Tomography", Med. Phys. 35 (4), Mar. 2008

Iterative Reconstruction: Parameters

- Image/object representation
  - Pixel centers
  - Pixel area
  - Blobs
  - Sampling density (pixel size, pixel locations, …)

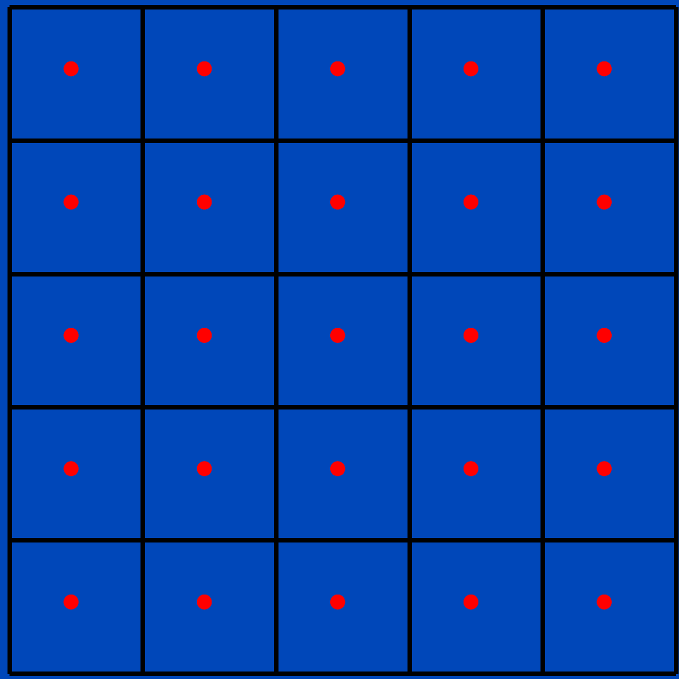
$$f(x,y)=\sum_m f_m b(x-x_m, y-y_m)$$

- Forward model (forward projection)
  - Joseph-type, Bresenham-type, distance-driven-type, …
  - Needle beam (infinitely thin ray), many needle beams per ray, …
  - Beam shape (varying beam cross-section, angular blurring, …)
  - Physical effects (beam hardening, scatter, motion, detector sensitivity, non-linear partial volume effect, …)
- Objective function, update equation
  - Statistical model (Gaussian, Poisson, shifted Poisson, …)
  - Regularisation (edge-preserving, …)
  - Artifact reduction

$$C(\boldsymbol{f})=\left(\boldsymbol{R}\cdot\boldsymbol{f}-\boldsymbol{p}\right)^2$$

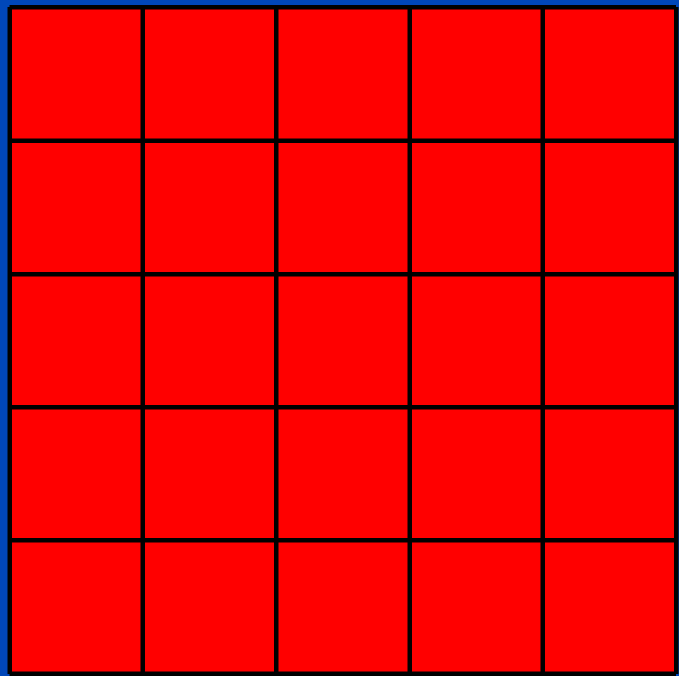
- Inverse model (backprojection)
  - Transpose of forward model
  - Pixel-driven backprojection
  - Filtered backprojection
  - …

Image Representation



$$b(x, y) = \bullet$$

Image Representation



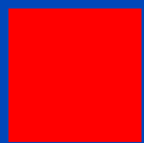
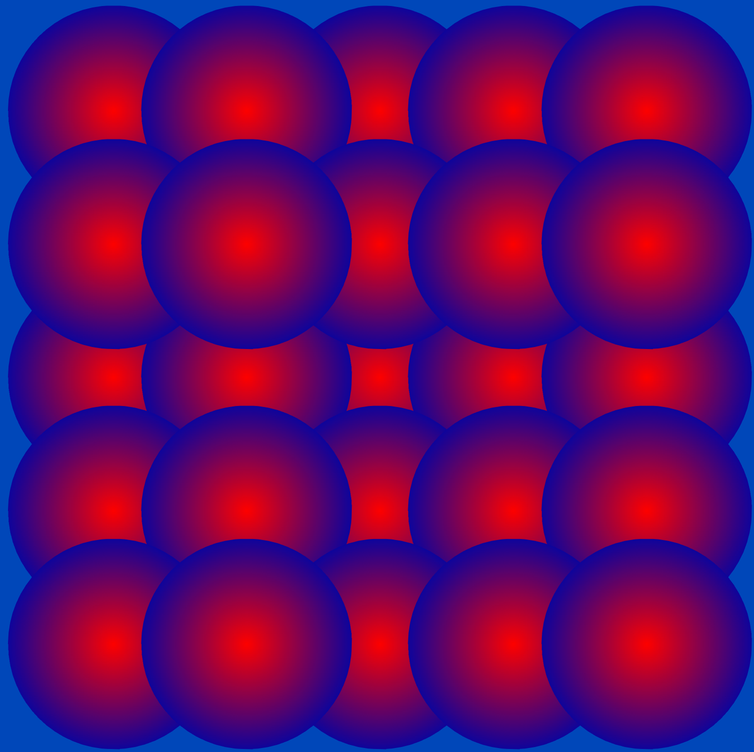
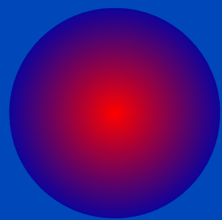
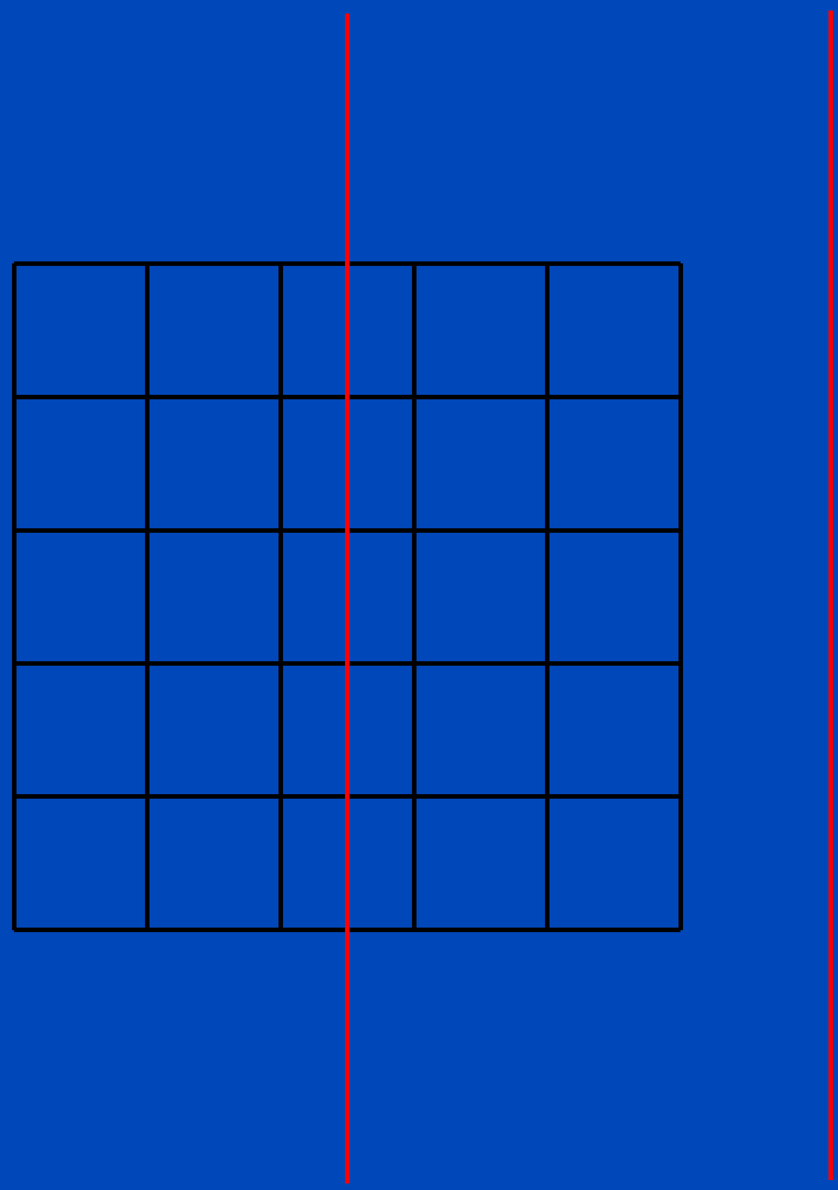
$b(x, y) =$ 

Image Representation

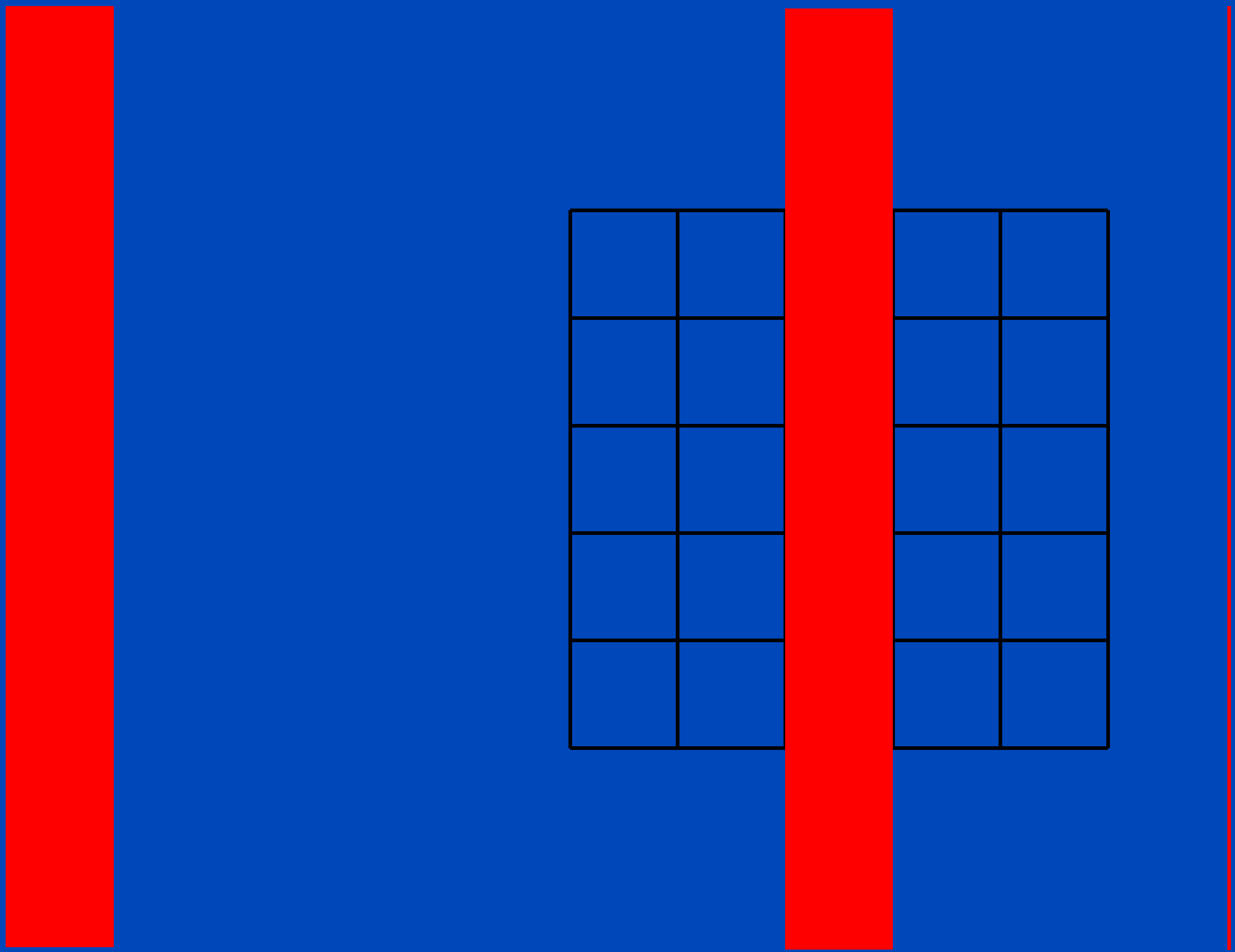


$b(x, y) =$ 

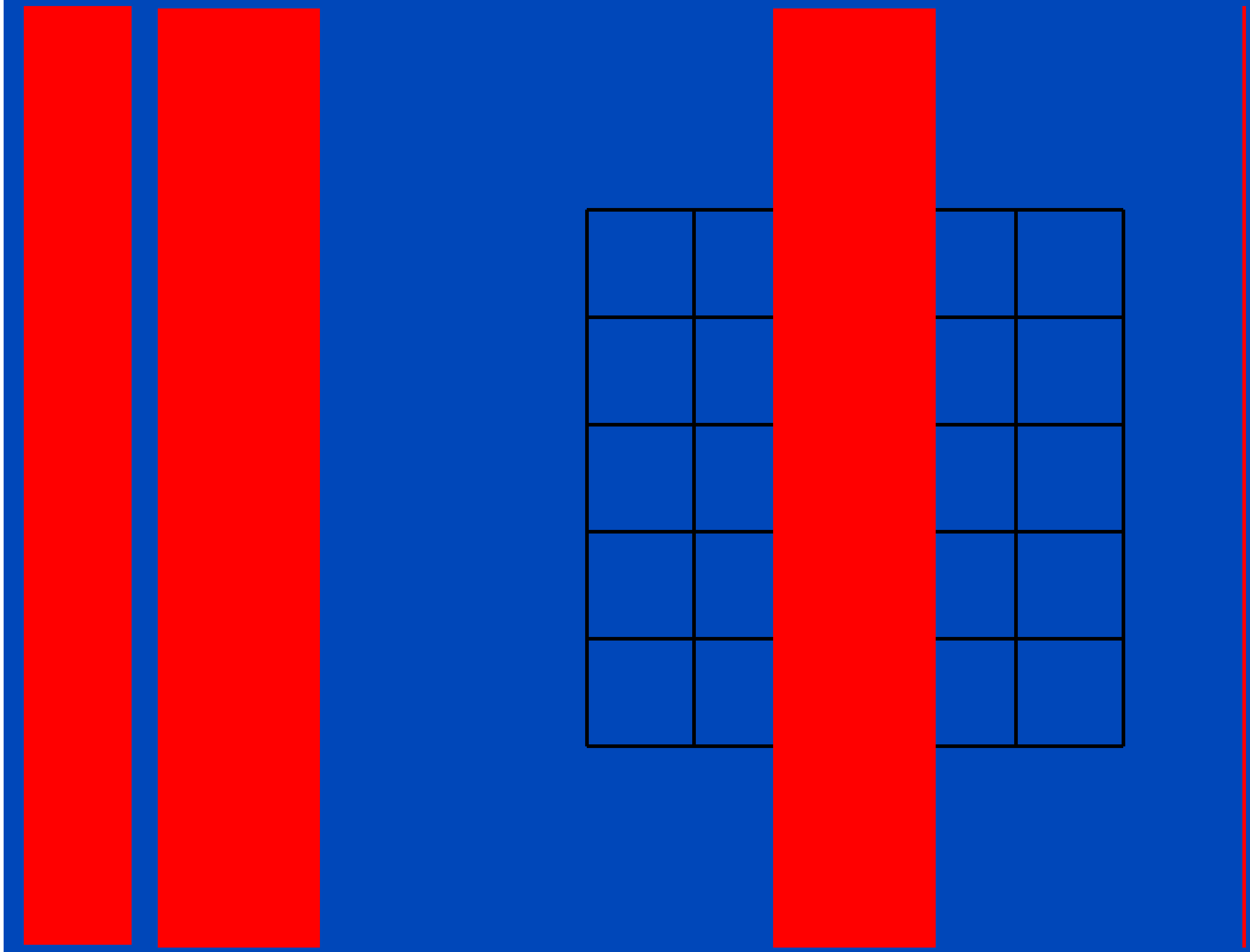
Forward Model: Beam Shape



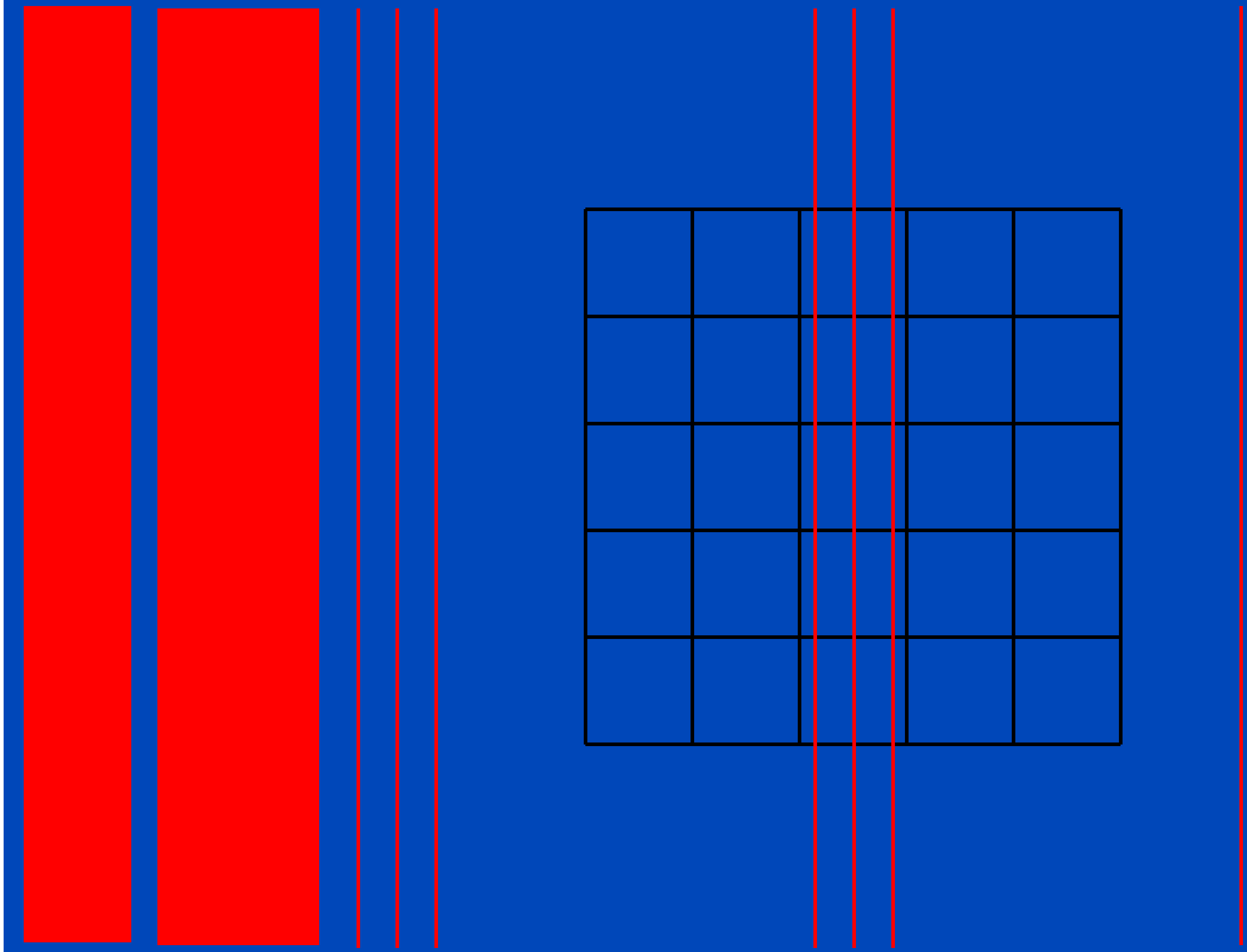
Forward Model: Beam Shape



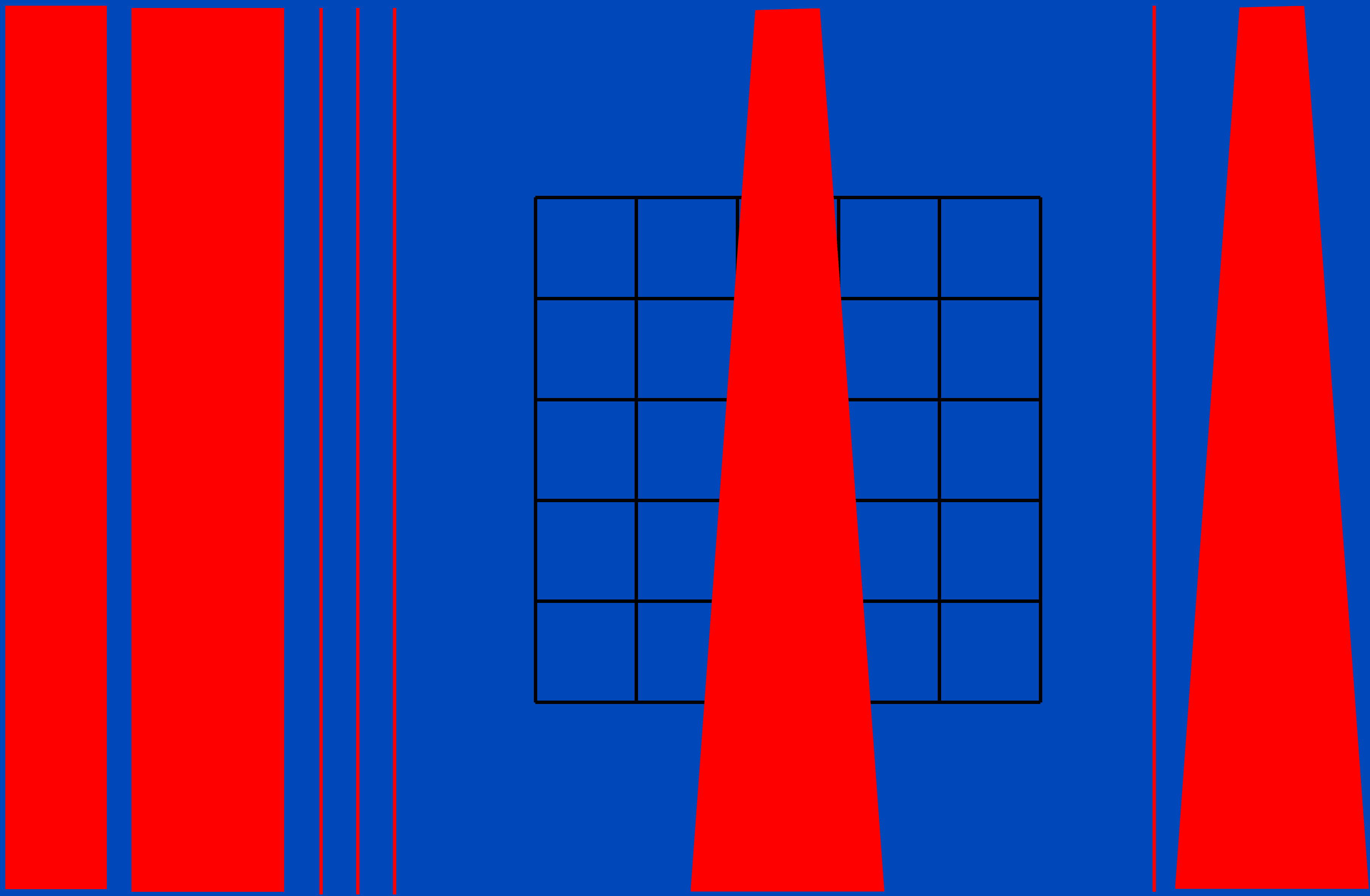
Forward Model: Beam Shape



Forward Model: Beam Shape



Forward Model: Beam Shape



Forward Model: Beam Shape

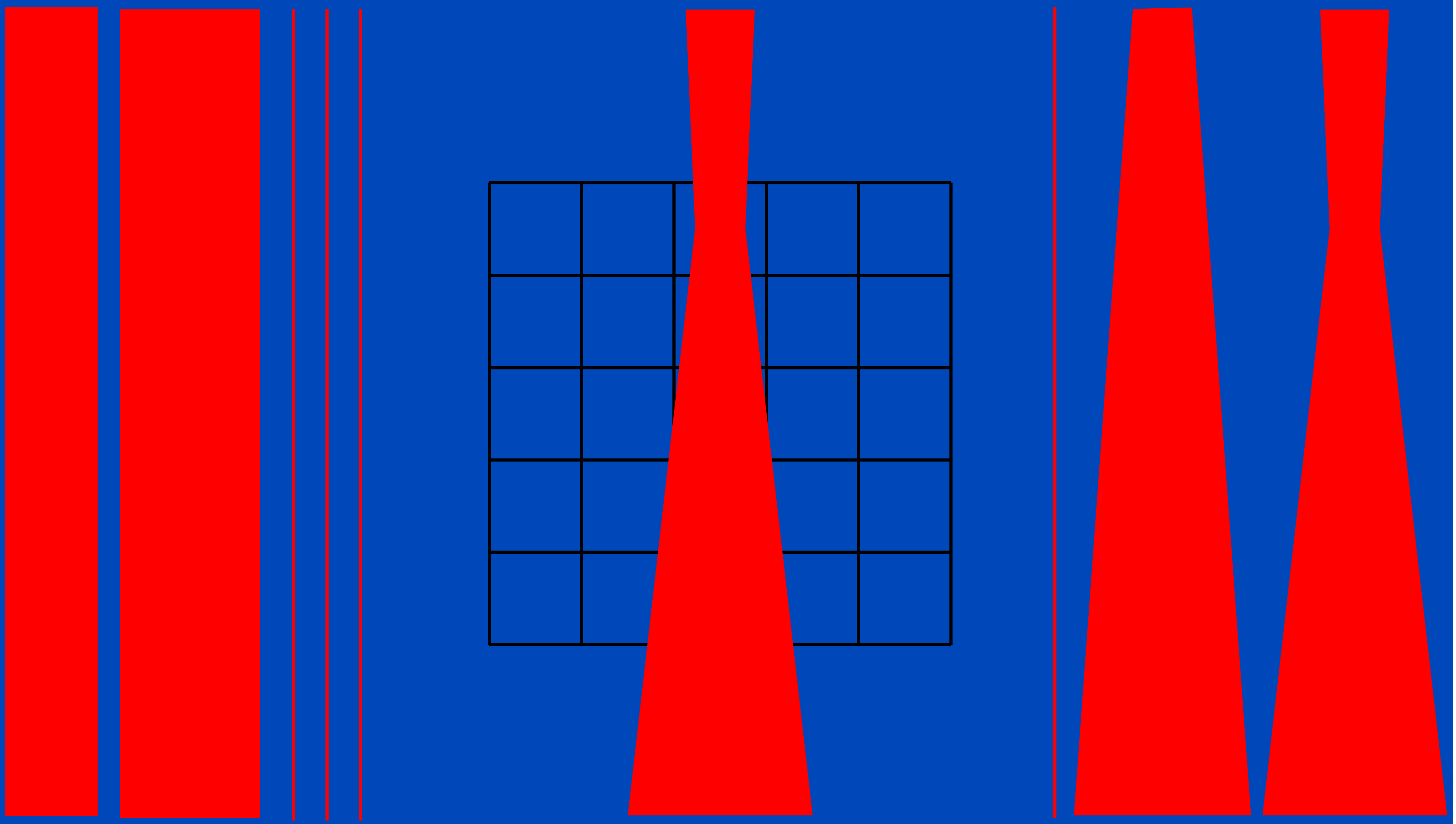
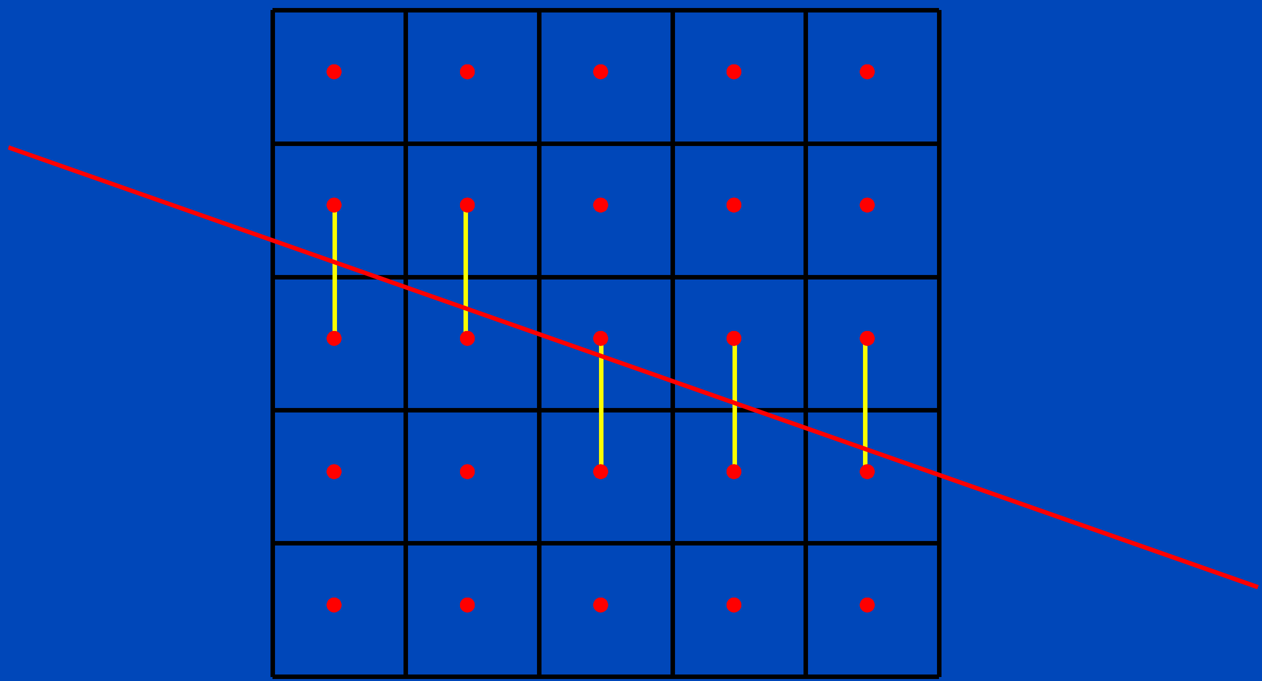


Image Representation and Forward Model are Linked!



Joseph's forward projector

Objective Function: Gauß Model

- **Assume that the attenuation is Gaussian-distributed**

$$\mathcal{L}(A) = \mathcal{N}(\sigma, \boldsymbol{r} \cdot \boldsymbol{f})$$

i.e. $P(A = a) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}(a-\mu)^2/\sigma^2}$ **with** $\mu = \boldsymbol{r} \cdot \boldsymbol{f}$ **.**

- **Consequently, the likelihood for all *N* measured signals is (** $\mu_n = \boldsymbol{r}_n \cdot \boldsymbol{f}$ **):**

$$P(\boldsymbol{A} = \boldsymbol{a}, \boldsymbol{f}) = \prod_n P(A_n = a_n)$$

- **Before maximizing take the log, penalize roughness,**

$$L(\boldsymbol{f}) = -\sum_n \left(\frac{a_n - \mu_n}{\sigma_n} \right)^2 - \beta R(\boldsymbol{f})$$

and then find the image *f* that maximizes *L*.

dkfz.

- **This leads us to minimizing**

$$(\boldsymbol{R}\cdot\boldsymbol{f}-\boldsymbol{a})^{\mathrm{T}}\cdot\boldsymbol{D}\cdot(\boldsymbol{R}\cdot\boldsymbol{f}-\boldsymbol{a})$$

which means solving

$$\boldsymbol{R}^{\mathrm{T}}\cdot\boldsymbol{D}\cdot(\boldsymbol{R}\cdot\boldsymbol{f}-\boldsymbol{a})=\boldsymbol{0}$$

- **This must be done numerically (e.g. Jacobi method) and the solutions are often of type**

$$\boldsymbol{f}_{\nu+1}=\boldsymbol{f}_{\nu}+\mathrm{diag}(\boldsymbol{u})\cdot\boldsymbol{R}^{\mathrm{T}}\cdot\mathrm{diag}(\boldsymbol{v})\cdot(\boldsymbol{a}-\boldsymbol{R}\cdot\boldsymbol{f}_{\nu})$$

Update Equation: Gauß Model

- **ART** $$f_{\nu+1} = f_\nu + R^{\mathrm{T}} \cdot \frac{p - R \cdot f_\nu}{R^2 \cdot 1}$$

- **SART** $$f_{\nu+1} = f_\nu + \frac{1}{R^{\mathrm{T}} \cdot 1} R^{\mathrm{T}} \cdot \frac{p - R \cdot f_\nu}{R \cdot 1}$$

- **and many more …**

Objective Function: Poisson Model

- **Assume that the intensities are Poisson-distributed**

$$\mathcal{L}(I) = \mathcal{P}(I_0 e^{-\boldsymbol{r}\cdot\boldsymbol{f}})$$

which means $P(I=i) = \frac{\mu^i}{i!}e^{-\mu}$ **with** $\mu = I_0 e^{-\boldsymbol{r}\cdot\boldsymbol{f}}$ **.**

- **Consequently, the likelihood for all *N* measured signals is (** $\mu_n = I_0 e^{-\boldsymbol{r}_n\cdot\boldsymbol{f}}$**):**

$$P(\boldsymbol{I}=\boldsymbol{i},\boldsymbol{f}) = \prod_n P(I_n = i_n) = \frac{\mu_n^{i_n}}{i_n!}e^{-\mu_n}$$

- **Before maximizing take the log, penalize roughness,**

$$L(\boldsymbol{f}) = \sum_n (i_n \ln \mu_n - \mu_n) - \beta R(\boldsymbol{f})$$

and then find the image *f* that maximizes *L*.

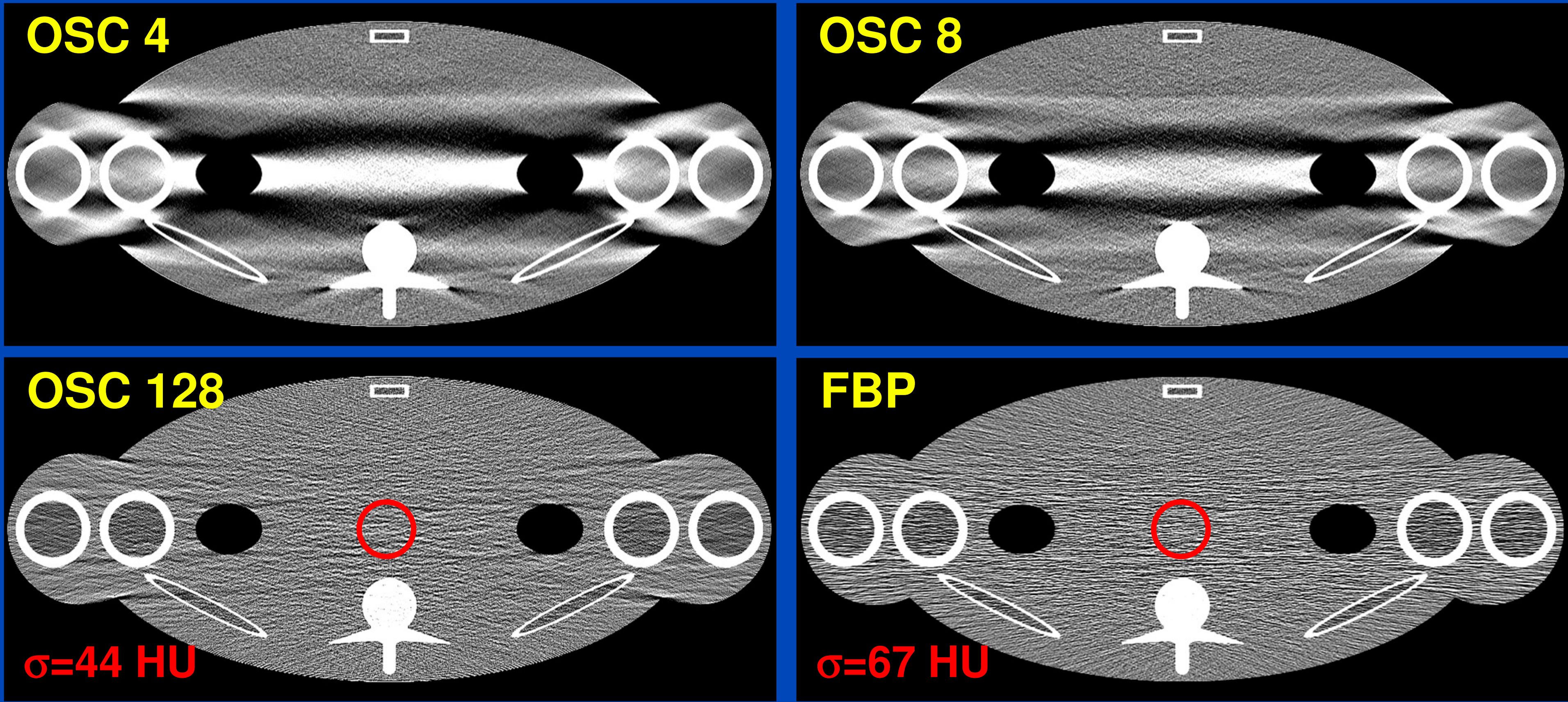
Update Equation: Poisson Model

- **MLEM** $$f_{\nu+1} = f_\nu \frac{\boldsymbol{R}^{\mathrm{T}} \cdot \left(e^{-\boldsymbol{R}\cdot \boldsymbol{f}_\nu}\right)}{\boldsymbol{R}^{\mathrm{T}} \cdot \left(e^{-\boldsymbol{p}}\right)}$$

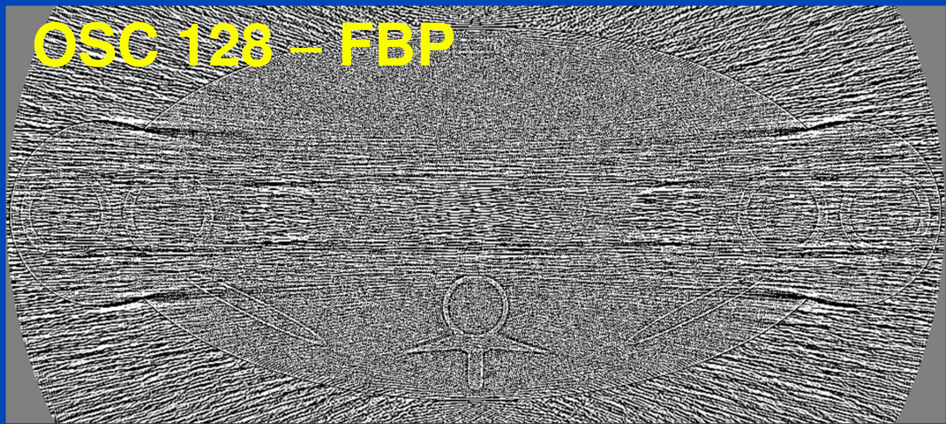
- **OSC** $$f_{\nu+1} = f_\nu + f_\nu \frac{\boldsymbol{R}^{\mathrm{T}} \cdot \left(e^{-\boldsymbol{R}\cdot \boldsymbol{f}_\nu} - e^{-\boldsymbol{p}}\right)}{\boldsymbol{R}^{\mathrm{T}} \cdot \left(e^{-\boldsymbol{R}\cdot \boldsymbol{f}_\nu} \boldsymbol{R} \cdot \boldsymbol{f}_\nu\right)}$$

- **and many more …**

Native OSC Converges Slowly



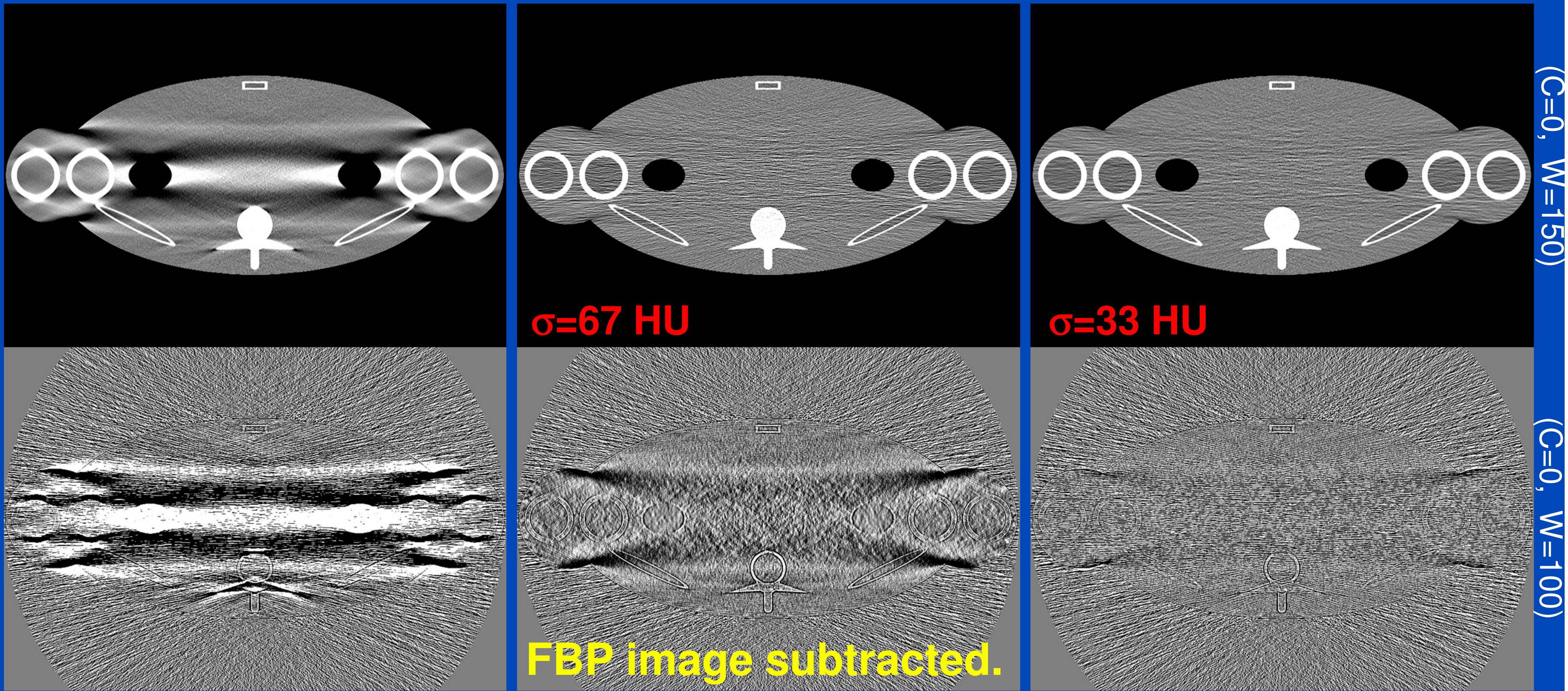
(C=0, W=150)



(C=0, W=100)

Proper Initialization Helps!

OSC 4, initialized with constant value	OSC 4, initialized with matched FBP	OSC 4, initialized with smooth FBP



Insufficient image quality	Same noise as FBP	50% less noise than FBP

Ordered Subsets

- Divide one iteration into S sub-iterations.
- Each of these S subsets covers N/S projections.
- During one iteration all subsets and therefore all projections are used exactly once.
- Per iteration the volume is updated S times (once per sub-iteration).
- An up to S-fold speed-up can be observed.

Ordered Subsets
Illustration for N = 32 Projections

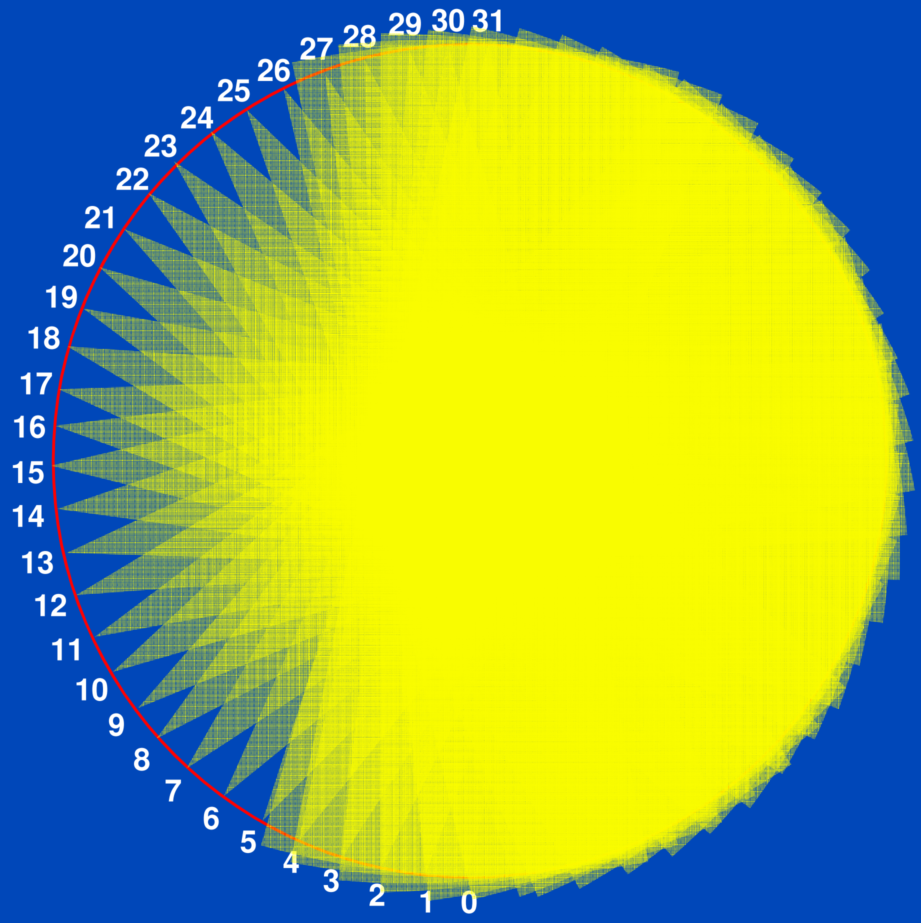
Conventional procedure without subets (S = 1)

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31

Ordered subsets with S = 8 sub-iterations

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31

Ordered Subsets

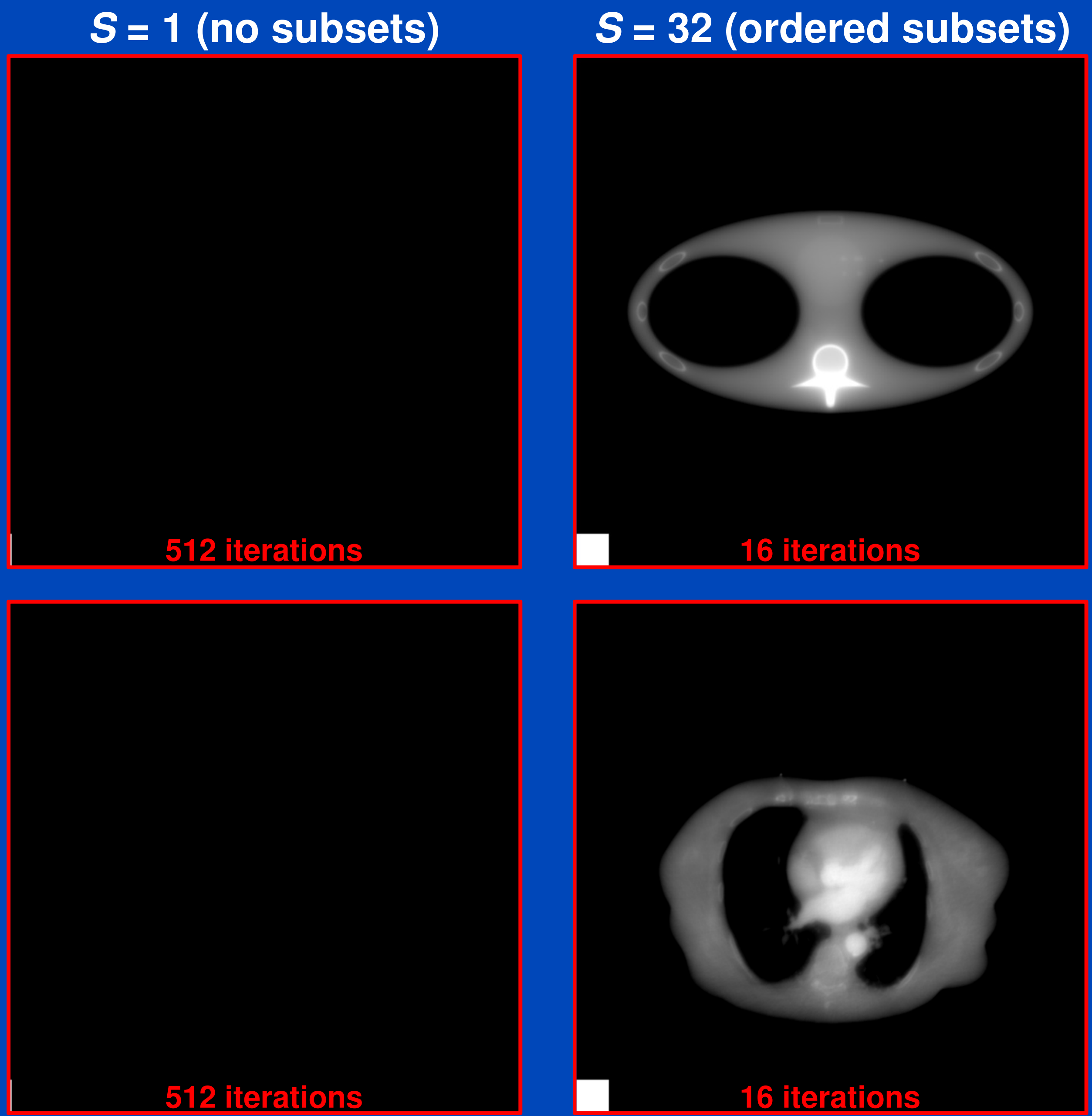


$N_{Projections}$ = 32, Ordered Subsets: $N_{Subsets}$ = 8

Sequence Can be Generated Using Simple Bit Reversal

0	->	0
1	->	16
2	->	8
3	->	24
4	->	4
5	->	20
6	->	12
7	->	28
8	->	2
9	->	18
10	->	10
11	->	26
12	->	6
13	->	22
14	->	14
15	->	30
16	->	1
17	->	17
18	->	9
19	->	25
20	->	5
21	->	21
22	->	13
23	->	29
24	->	3
25	->	19
26	->	11
27	->	27
28	->	7
29	->	23
30	->	15
31	->	31

Iterations



C = 0 HU, W = 1000 HU

Image Updates

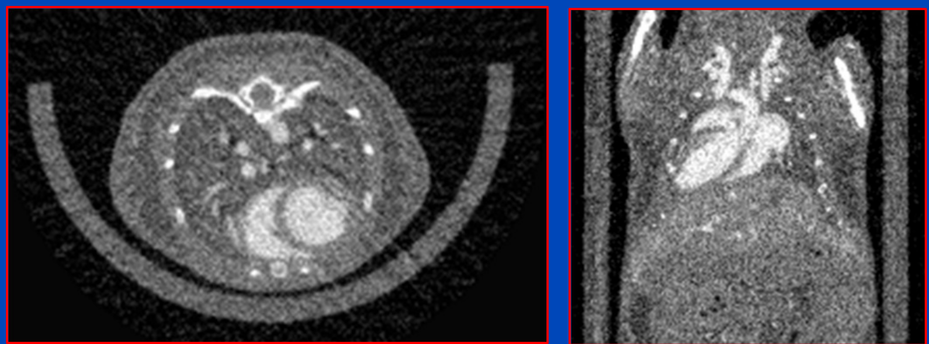
S = 1 (no subsets)	*S* = 32 (ordered subsets)
	
	

C = 0 HU, *W* = 1000 HU

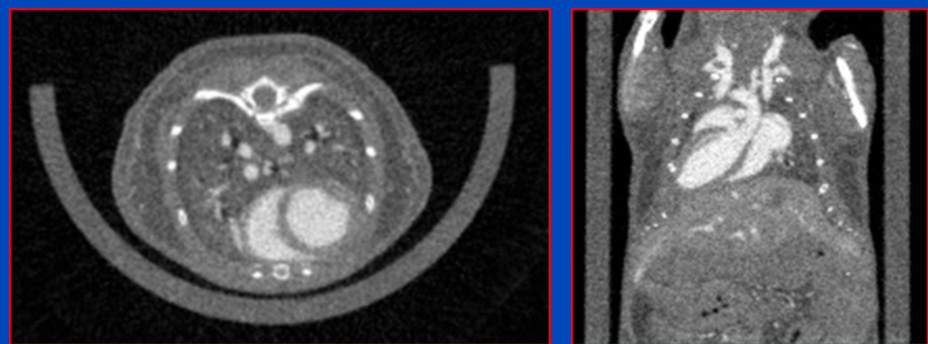
What Makes Iterative Recon Attractive?

- No need to come find an analytical solution
- Works for all geometries with only small adaptations
- Allows to model any effect
- Allows to incorporate prior knowledge
  - noise properties (quantum noise, electronic noise, noise texture, …)
  - prior scans (e.g. planning CT, full scan data, …)
  - image properties such as smoothness, edges (e.g. minimum TV)
  - …
- Handles missing data implicitly (but not necessarily better)

Phase-correlated Feldkamp



High dimensional TV minimization[1]



[1]L. Ritschl, S. Sawall, M. Knaup, A. Hess, and M. Kachelrieß, Phys. Med. Biol. 57, Jan. 2012

IOP PUBLISHING INVERSE PROBLEMS

Inverse Problems **25** (2009) 123009 (36pp) doi:10.1088/0266-5611/25/12/123009

TOPICAL REVIEW

Why do commercial CT scanners still employ traditional, filtered back-projection for image reconstruction?

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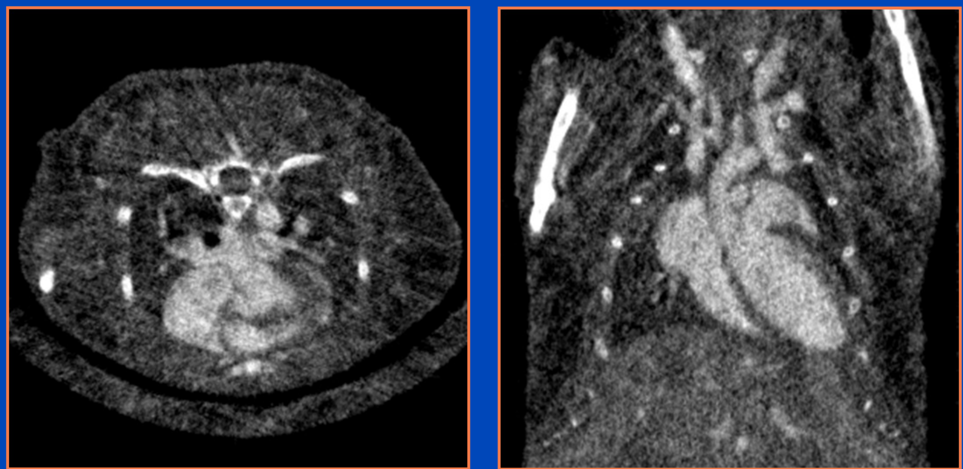
Abstract

Despite major advances in x-ray sources, detector arrays, gantry mechanical design and especially computer performance, one component of computed tomography (CT) scanners has remained virtually constant for the past

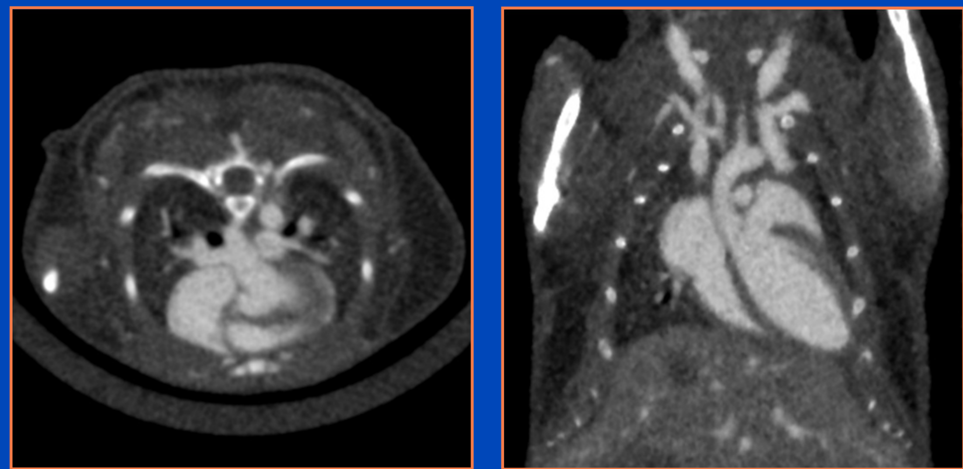
Iterative != Iterative

- **In many cases artifact correction is iterative**
  - Higher order beam hardening correction
  - Cone-beam artifact correction
  - Scatter correction
- **Practical "iterative reconstruction" approaches**
  - often use empirical solutions
  - combine iterative with analytical reconstruction
  - combine iterative or analytical reconstruction with image restoration

Phase-correlated Feldkamp



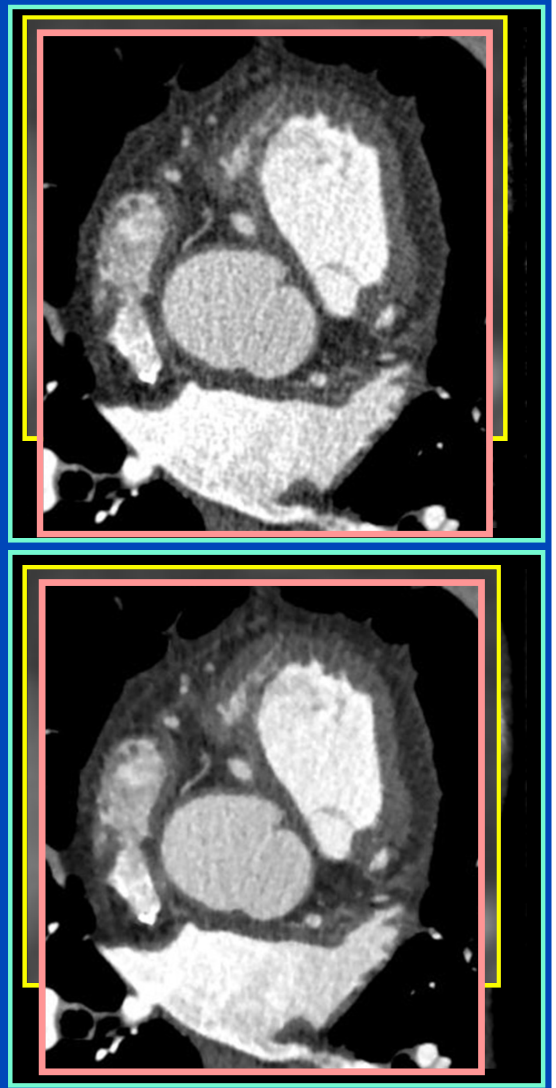
Low dose phase-correlated (LDPC) recon[1]



[1]S. Sawall, F. Bergner, R. Lapp, M. Mronz, A. Hess, and M. Kachelrieß, MedPhys 38(3), 2011

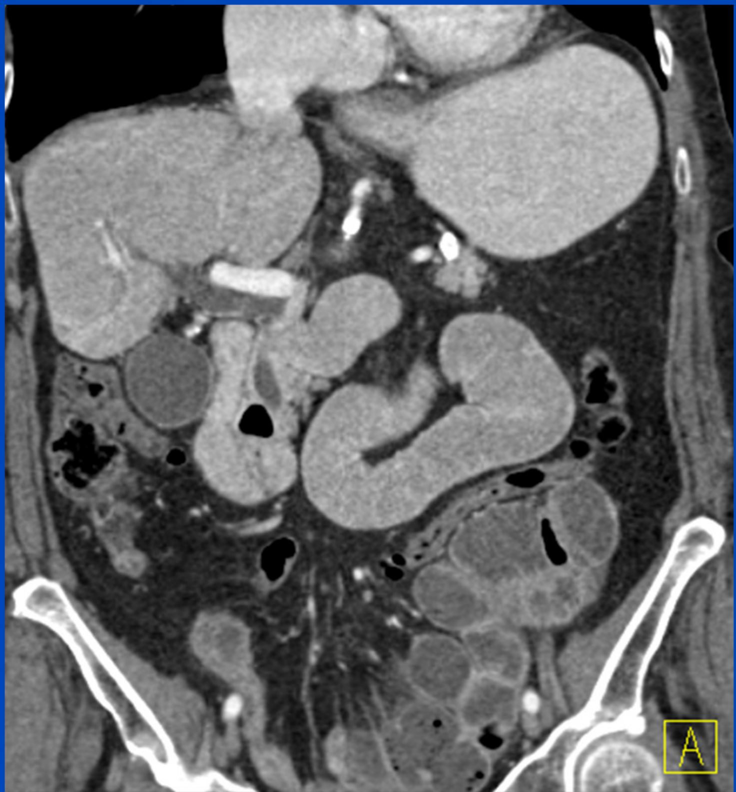
Iterative Reconstruction

- Aim: less artifacts, lower noise, lower dose
- Iterative reconstruction
  - Reconstruct an image.
  - Regularize the image.
  - Does the image correspond to the rawdata?
  - If not, reconstruct a correction image and continue.
- SPECT + PET are iterative for a long time.
- Until recently, the computational demand prohibited to use iterative recon in CT.
- First CT product implementations
  - AIDR (adaptive iterative dose recuction, Toshiba)
  - ASIR (adaptive statistical iterative reconstruction, GE)
  - iDose (Philips)
  - IRIS (image reconstruction in image space, Siemens)
  - VEO, MBIR (model-based iterative reconstruction, GE)
  - SAFIRE (sinogram-affirmed iterative reconstruction, Siemens)



Conventional reconstruction

at 100% dose



Iterative reconstruction and restoration

at 40% dose

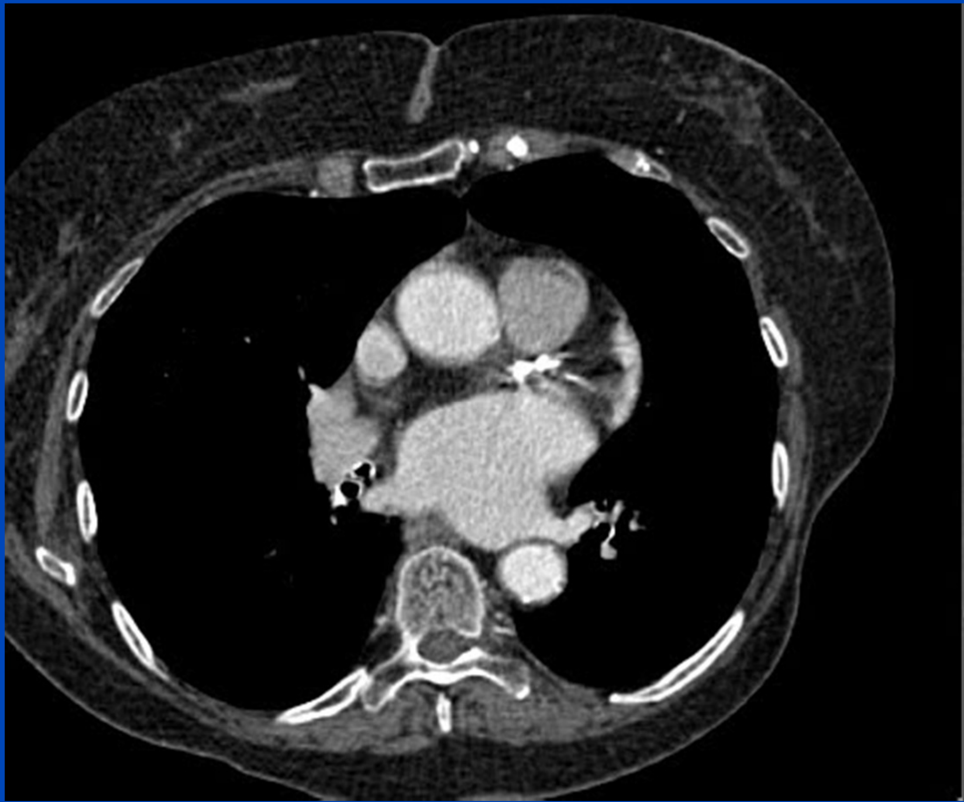
Images provided by Siemens Healthcare, Forchheim, Germany

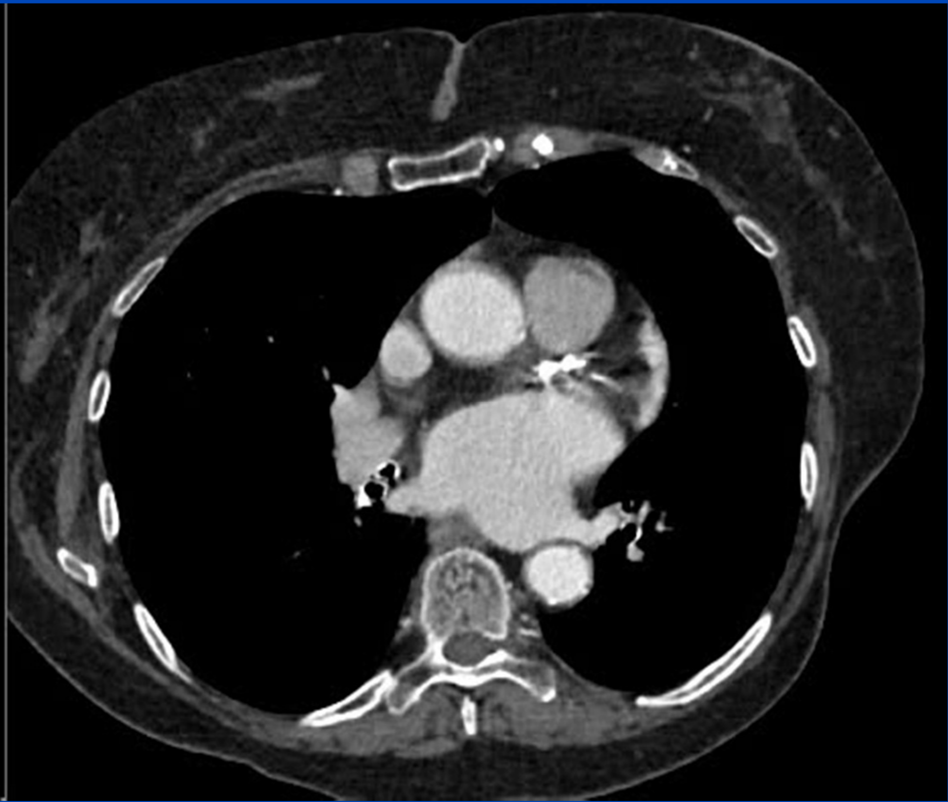
Conventional reconstruction

at 100% dose

Iterative reconstruction and restoration

at 40% dose

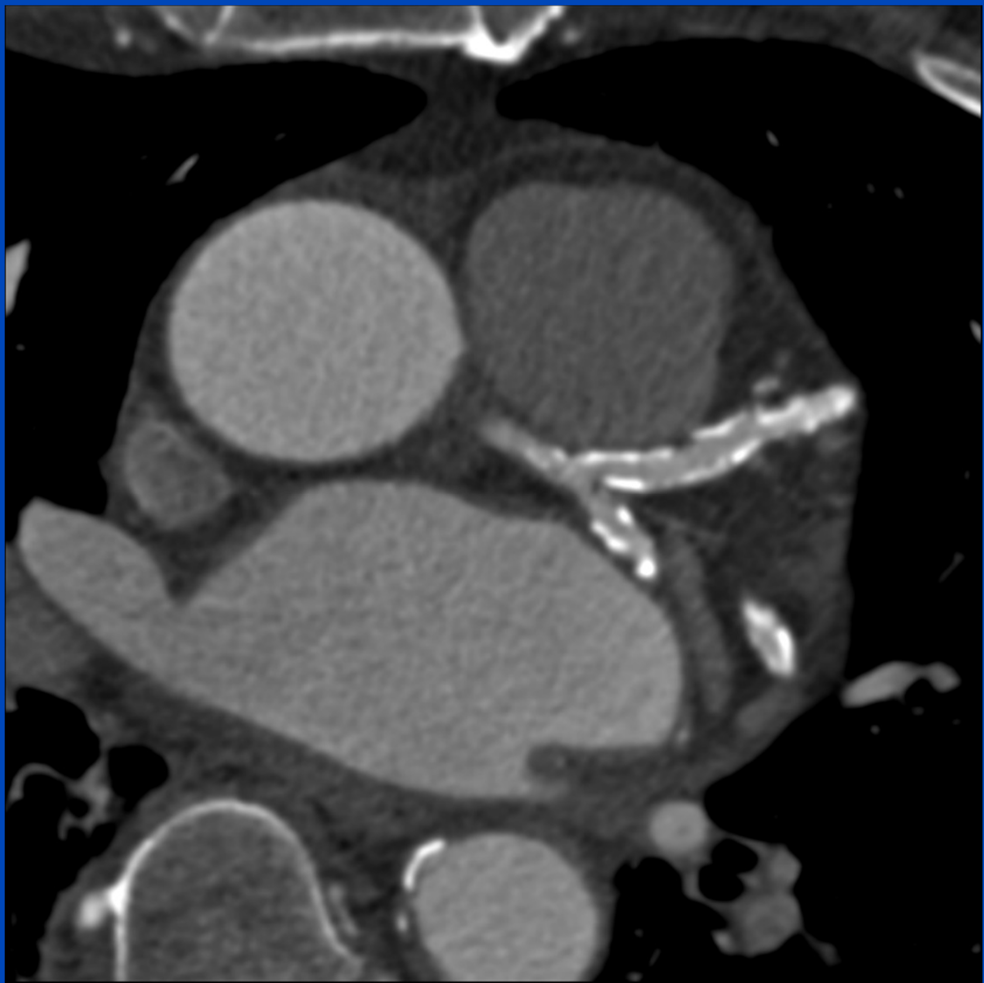




Images provided by Siemens Healthcare, Forchheim, Germany

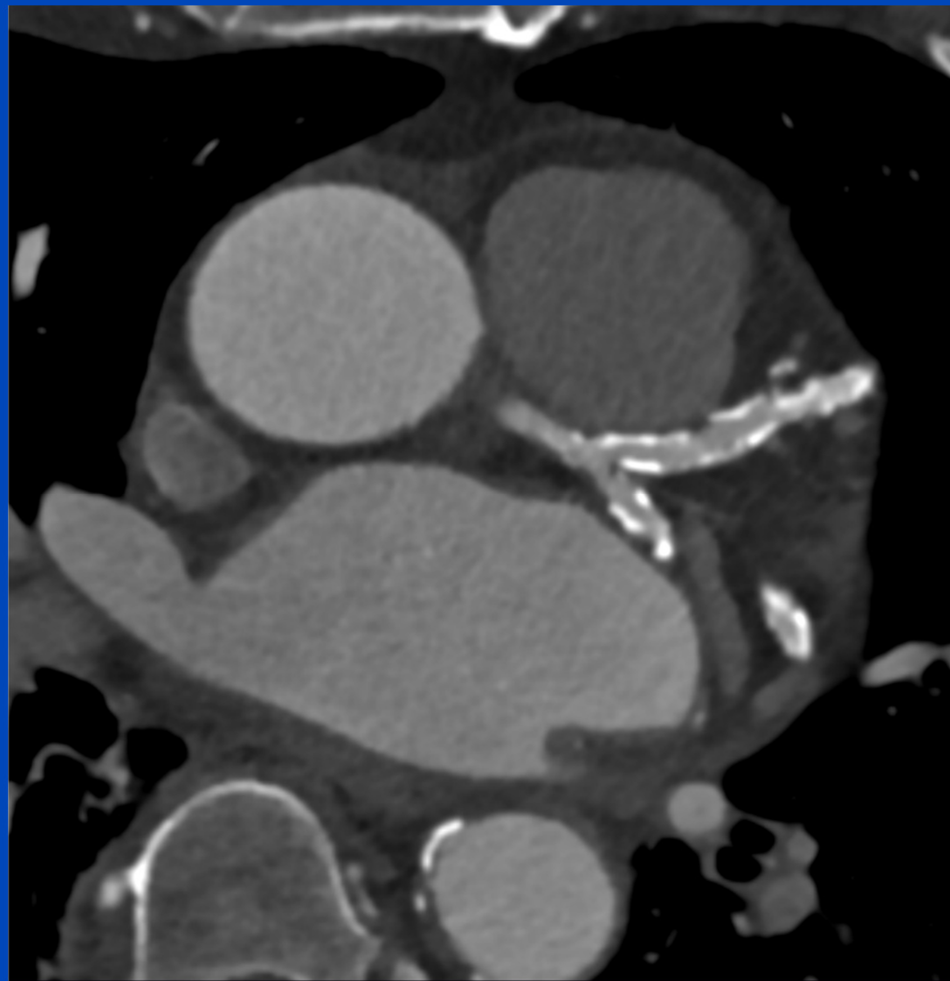
Conventional reconstruction

at 100% dose



Iterative reconstruction and restoration

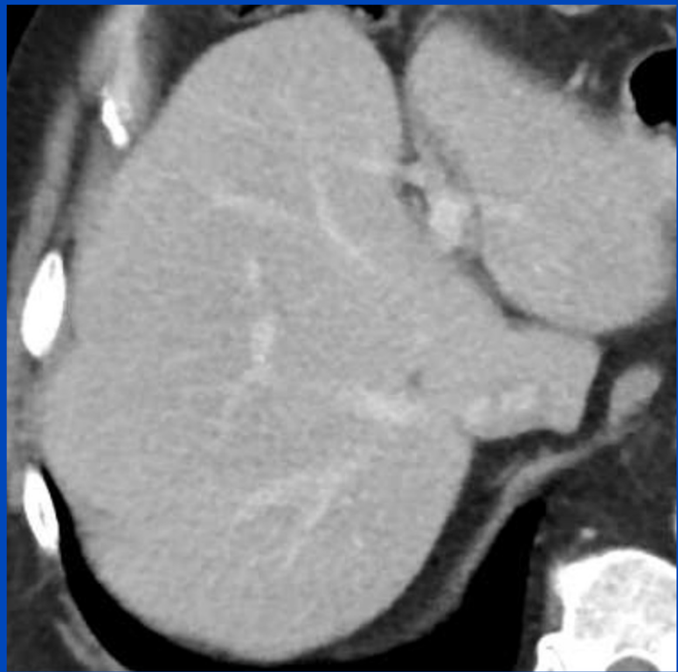
at 40% dose



Images provided by Siemens Healthcare, Forchheim, Germany

100% dose

50% dose

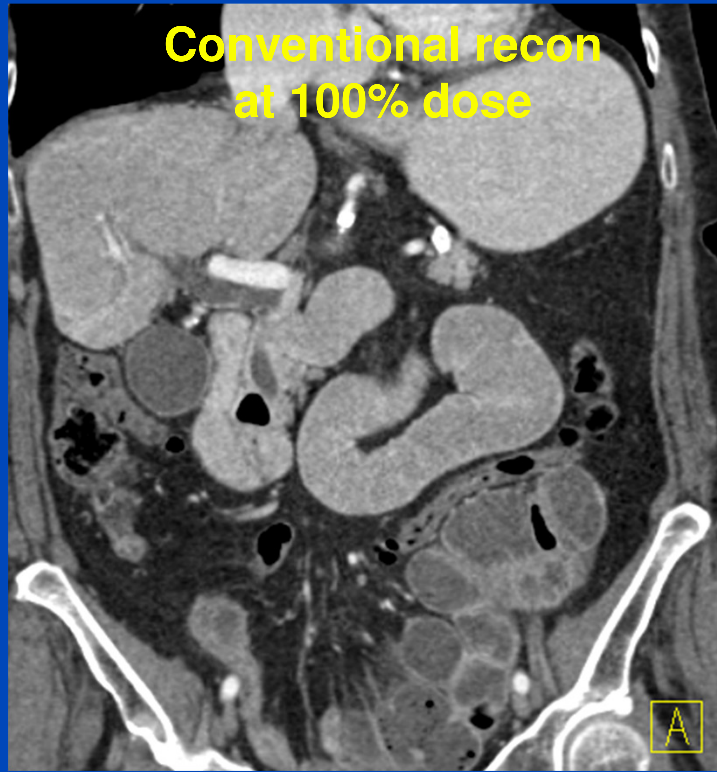


50% dose + IRIS

Image courtesy of Prof. Dr. Michael Lell, Erlangen, Germany

Summary

- **Analytical image reconstruction**
  - is compute efficient
  - requires new solutions for new trajectories
  - is what most images are reconstructed with
- **Iterative image reconstruction**
  - requires much more computational effort
  - allows to easily model constraints
  - allows to incorporate prior knowledge
- **Practical modern solutions**
  - often are a combination of analytical and iterative recon
  - are offered by the major manufacturers of diagnostic CT



Images provided by Siemens Healthcare, Forchheim, Germany


Thank You!
This presentation will soon be available at www.dkfz.de/ct.
The iteration videos were prepared by my colleague Christian Hofmann.